

## ON MODULES OF FINITE PROJECTIVE DIMENSION

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ABSTRACT. We address two aspects of finitely generated modules of finite projective dimension over local rings and their connection in between: embeddability and grade of order ideals of minimal generators of syzygies. We provide a solution of the embeddability problem and prove important reductions and special cases of the order ideal conjecture. In particular we derive that in any local ring  $R$  of mixed characteristic  $p > 0$ , where  $p$  is a non-zero-divisor, if  $I$  is an ideal of finite projective dimension over  $R$  and  $p \in I$  or  $p$  is a non-zero-divisor on  $R/I$ , then every minimal generator of  $I$  is a non-zero-divisor. Hence if  $P$  is a prime ideal of finite projective dimension in a local ring  $R$ , then every minimal generator of  $P$  is a non-zero-divisor in  $R$ .

In this note we would like to consider two aspects of finitely generated modules of finite projective dimension over any local ring: embeddability and grade of order ideals of minimal generators of its syzygies (minimal). In regard to embeddability Auslander and Buchweitz ([A-B]) proved that any finitely generated module on a Gorenstein local ring can be embedded in a module of finite projective dimension such that the cokernel is Cohen-Macaulay. This result has several applications in solving homological questions and conjectures in commutative algebra including Serre's  $\chi_i$ -conjectures and its generalizations on intersection multiplicity ([D3]). For all these conjectures one is usually concerned with finitely generated modules  $M$  which are of finite projective dimension over a local ring  $R$ , but are not so over  $R/xR$ ,  $x$  being a non-zero-divisor in the annihilator of  $M$  in  $R$ . However, a similar result has been absent for non-Gorenstein local rings until now. In this note we prove the following with respect to embeddability for modules of finite projective dimension:

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**Theorem 1.2.** *Let  $(R, m)$  be a local ring and let  $M$  be a finitely generated module of finite projective dimension with positive grade over  $R$ . Let  $\mathbf{x} = \{x_1, \dots, x_t\}$  be any  $R$ -sequence contained in the annihilator of  $M$  (henceforth  $\text{ann}_R M$ ) and let  $\overline{R} = R/(x_1, \dots, x_t)$ . Then there exists a short exact sequence of finitely generated  $\overline{R}$ -modules*

$$0 \rightarrow M \rightarrow Q \rightarrow T \rightarrow 0$$

where  $\text{pd}_{\overline{R}} Q < \infty$  and  $\text{pd}_R T = t$ .

From the construction of  $Q$  it would follow that support of  $Q$  over  $\overline{R} = \text{Spec}(\overline{R})$  and  $Q$  satisfies the strong intersection conjecture due to Peskine and Szpiro ([P-S1]) (Remark 1.2). Since  $T$  is perfect,  $T$  also possesses the above property. The effect of this theorem on generalizations of Serre's conjectures on intersection multiplicities for arbitrary local rings, in particular for Cohen-Macaulay rings, would be the subject matter of a future paper. In this paper we focus on its effect on the order ideal conjecture which is our next topic.

The order ideal conjecture stems from Evans and Griffith's work on grade of order ideals of minimal generators of syzygies in equicharacteristic. The statement of the conjecture is the following.

**Order Ideal Conjecture.** *Let  $(R, m)$  be a local ring. Let  $M$  be a finitely generated module of finite projective dimension over  $R$  and let  $S_i$  denote its  $i^{\text{th}}$  syzygy for  $i > 0$ . If  $\beta$  is a minimal generator of  $S_i$ , then the order ideal  $\mathcal{O}_{S_i}(\beta)$  has grade at least  $i$ .*

Let us recall that  $\mathcal{O}_{S_i}(\beta) = \{f(\beta) | f \in \text{Hom}_R(S_i, R)\}$ .

We say that a module  $M$  satisfies the order ideal conjecture if order ideals of minimal generators of all its syzygies satisfy the respective grade inequalities mentioned above.

Evans and Griffith ([E-G1], [E-G2]) proved the above conjecture for equicharacteristic local rings in order to solve the syzygy problem over the above class of rings. The existence of big Cohen-Macaulay modules, due to Hochster ([H1]), played an important role in their proof. Later they proved a graded version of the above conjecture for a certain class of graded rings in mixed characteristic ([E-G3]). We would also refer the reader to theorem 9.5.2 in [Br-H] for a more general version of the order ideal theorem in the equicharacteristic case. Actually for their proof of the syzygy theorem in equicharacteristic Evans and Griffith only needed to prove the above conjecture for modules which are locally

free on the punctured spectrum of  $\text{spec}(R)$ ,  $R$  being regular local. And they reduced the proof of this case to what is now known as improved new intersection conjecture. Later Hochster showed that the canonical element conjecture implies the improved new intersection conjecture [H2]. The equivalence of these two conjectures was established in [D1]. In our most recent work [D2] we have shown that *a particular case of the order ideal conjecture implies the monomial conjecture* and hence all its equivalent forms, e.g., the direct summand conjecture, the canonical element conjecture, the improved intersection conjecture etc. Thus the order ideal conjecture now occupies a central position among several homological conjectures in commutative algebra.

First let us mention that in order to prove the order ideal conjecture on arbitrary local rings  $R$ , *it is enough to concentrate on first syzygies* of modules of finite projective dimension (Lemma 2.1). Theorem 2.3 shows that *for the validity of the order ideal conjecture it is enough to prove that every minimal generator of ideals of height 2, grade 2 and of finite projective dimension over  $R$  is a non-zero-divisor*. In theorem 2.5 we prove the following:

*Given a module of finite projective dimension  $M$  on a local ring  $R$ , there exists an  $R$ -sequence  $x_1, \dots, x_h$ ,  $h = \text{pd}_R M$ , such that for any  $R$ -module  $N$  for which  $x_1, \dots, x_h$  form an  $N$ -sequence,  $\text{Tor}_j^R(M, N) = 0$  for  $j > 0$ .*

This theorem leads us to the following (Corollary 2, 2.7):

*Let  $(R, m)$  be a local ring and  $M$  be a finitely generated module of finite projective dimension over  $R$ . Let  $x_1, \dots, x_h$  be an  $R$ -sequence as mentioned in the above proposition. If  $R$  has mixed characteristic  $p > 0$ , we assume that  $p, x_1, \dots, x_h$  form a part of a system of parameters of  $R$ . Then for every minimal generator  $\beta$  of  $S_1 = \text{Syz}_R^1(M)$ ,  $\text{grade } \mathcal{O}_{S_1}(\beta) \geq 1$ .*

The statement of our main theorem in section 2 is the following.

**Theorem 2.11.** *Let  $(R, m)$  be a local ring of mixed characteristic  $p > 0$ . Let  $M$  be a finitely generated module of finite projective dimension over  $R$  and let  $\beta$  be a minimal generator of  $S_i$ , the  $i^{\text{th}}$  syzygy (minimal) of  $M$ , for  $i > 0$ . We assume that either  $p$  is a non-zero-divisor in  $R$  or  $p$  is nilpotent. We have the following:*

- a) *if  $p$  is nilpotent, then the order ideal conjecture is valid on  $R$ .*
- b) *if  $pM = 0$ , then  $\text{grade of } \mathcal{O}_{S_1}(\beta) \geq 1$ .*
- c) *Suppose that balanced or complete almost Cohen-Macaulay algebras exist over com-*

plete local domains. If  $pM = 0$  and  $p \in m - m^2$ , then  $\text{grade } \mathcal{O}_{S_i}(\beta) \geq i, \forall i \geq 1$ .

- d) Assume that every element in  $m - m^2$  is a non-zero-divisor and the order ideal conjecture is valid over  $R/xR$  for  $x \in m - m^2$ . If  $\text{ann}_R M \cap (m - m^2) \neq \phi$  or  $\text{depth}_R M > 0$ , then  $\text{grade } \mathcal{O}_{S_i}(\beta) \geq i, \forall i \geq 1$ .

As a consequence of the above theorem we have the following corollaries:

**Corollary 2.12.** *For any ideal  $I$  of finite projective dimension over  $R$  of mixed characteristic  $p > 0$ , where  $p$  is a non-zero-divisor in  $R$ , if  $p \in I$  or  $p$  is a non-zero-divisor on  $R/I$  then every minimal generator of  $I$  is a non-zero-divisor. In particular if  $P$  is a prime ideal of finite projective dimension over  $R$ , then every minimal generator of  $P$  is a non-zero-divisor in  $R$ .*

**Corollary 2.13.** *Let  $(R, m)$  be a regular local ring of dimension  $n$  and assume that the order ideal conjecture is valid for regular local rings of dimension  $(n - 1)$ . If  $M$  is a finitely generated  $R$ -module such that either  $M$  is annihilated by a regular parameter or  $\text{depth}_R M > 0$ , then  $M$  satisfies the order ideal conjecture.*

The two main ingredients of our proof of theorem (2.11) are theorem 1.2 and Shimomoto's theorem (2.10) on existence of almost Cohen-Macaulay algebras ([Shi]).

Throughout this note "local" means noetherian local. For definitions of standard notions like projective dimension, grade etc. and their basic properties we refer the reader to [Br-H].

## SECTION 1

First we would like to mention the following proposition.

**1.1 Proposition** (Prop. 1.1, [D1]). *Let  $A$  be a noetherian local ring and let  $F_\bullet : \rightarrow A^{s_i} \rightarrow A^{s_{i-1}} \rightarrow \dots \rightarrow A^{s_0} \rightarrow 0$  be a free complex with  $H_0(F_\bullet) = M$ . Let  $N \subset M$  be a submodule of  $M$ . Then we can construct a free complex  $G_\bullet : \rightarrow A^{d_i} \rightarrow A^{d_{i-1}} \rightarrow \dots \rightarrow A^{d_0} \rightarrow 0$  with  $H_0(G) = N$  and a map  $\phi_\bullet : G_\bullet \rightarrow F_\bullet$  such that  $\phi_0$  induces the inclusion  $N \hookrightarrow M$  and the mapping cone of  $\phi_\bullet$  is a free resolution of  $M/N$  i.e.,  $\phi_\bullet$  induces an isomorphism  $H_i(G_\bullet) \xrightarrow{\sim} H_i(F_\bullet)$  for  $i > 0$ . Moreover, by our construction,  $G_\bullet$  is minimal.*

For a proof we refer the reader to Prop. 1.1 in [D1].

Next we prove the main theorem of this section.

**1.2. Theorem.** *Let  $(R, m)$  be a local ring and let  $M$  be a finitely generated module of finite projective dimension over  $R$ . Assume that  $\text{grade}_R M > 0$ . Let  $\mathbf{x}$  denote the ideal generated by an  $R$ -sequence of length  $t$  contained in  $\text{ann}_R M$  and let  $\overline{R} = R/\mathbf{x}R$ . Then there exists a short exact sequence of finitely generated  $\overline{R}$ -modules*

$$0 \rightarrow M \rightarrow Q \rightarrow T \rightarrow 0$$

such that  $\text{pd}_{\overline{R}} Q < \infty$  and  $\text{pd}_R T = t$ .

*Proof.* First suppose that  $\text{pd}_R M = 1$ . Let us recall that  $\text{grade}_R M \leq \text{pd}_R M$ . Since  $\text{grade}_R M > 0$ , it follows that  $\text{grade}_R M = 1$ . Let  $x \in \text{ann}_R M$  be a non-zero-divisor. Consider a minimal free resolution:  $0 \rightarrow R^{t_1} \xrightarrow{f} R^{t_0} \rightarrow M \rightarrow 0$  of  $M$  over  $R$ . Tensoring this resolution with  $R/xR$  we obtain an exact sequence:  $0 \rightarrow \text{Tor}_1^R(M, R/xR) \rightarrow (R/xR)^{t_1} \xrightarrow{\bar{f}} (R/xR)^{t_0} \rightarrow M \rightarrow 0$ . Since  $x \in \text{ann}_R M$ ,  $\text{Tor}_1^R(M, R/xR) \simeq M$ . Let  $T = \text{im } \bar{f}$ . Hence we obtain the following short exact sequence:  $0 \rightarrow M \rightarrow (R/xR)^{t_1} \rightarrow T \rightarrow 0$  and our assertion follows. So we can assume that  $\text{pd}_R M \geq 2$ .

Let  $(F_\bullet, d_\bullet) : 0 \rightarrow R^{r_n} \xrightarrow{d_n} R^{r_{n-1}} \rightarrow \dots \rightarrow R^{r_1} \xrightarrow{d_1} R^{r_0} \rightarrow 0$  be a minimal projective resolution of  $M$  over  $R$  and let  $(L_\bullet, \phi_\bullet) : \overline{R}^{s_n} \xrightarrow{\phi_n} \overline{R}^{s_{n-1}} \rightarrow \dots \rightarrow \overline{R}^{s_1} \xrightarrow{\phi_0} \overline{R}^{s_0} \rightarrow 0$  be a minimal projective resolution of  $M$  over  $\overline{R}$ . Since  $\text{pd}_R M = n$ ,  $\text{grade Ext}_R^i(M, R) \geq i$  for  $1 \leq i \leq n$ ,  $\text{Ext}_R^n(M, R) \neq 0$  and  $\text{Ext}_R^i(M, R) = 0$  for  $i > n$ . Hence  $\text{grade Ext}_{\overline{R}}^i(M, \overline{R}) \geq i$  for  $1 \leq i \leq n - t$ ,  $\text{Ext}_{\overline{R}}^{n-t}(M, \overline{R}) \neq 0$  ( $\simeq \text{Ext}_R^n(M, R)$ ) and  $\text{Ext}_{\overline{R}}^i(M, \overline{R}) = 0$  for  $i > (n - t)$ .

Applying  $\text{Hom}_{\overline{R}}(-, \overline{R})$  to  $L_\bullet$  we obtain the following free complex  $L_\bullet^*$ :

$$L_\bullet^* : 0 \rightarrow \overline{R}^{s_0^*} \rightarrow \overline{R}^{s_1^*} \rightarrow \dots \rightarrow \overline{R}^{s_{n-t}^*} \rightarrow 0.$$

(For any  $R(\overline{R})$  module  $V$ ,  $V^* = \text{Hom}_R(M, R)(\text{Hom}_{\overline{R}}(V, \overline{R}))$ )

Let  $G = \text{Coker } \phi_{n-t}^*$ . We have a short exact sequence

$$0 \rightarrow \text{Ext}_{\overline{R}}^{n-t}(M, \overline{R}) \xrightarrow{c} G \xrightarrow{\eta} \text{Im } \phi_{n-t+1}^* \rightarrow 0. \quad (1)$$

By the above proposition, there exists a minimal free complex  $(P_\bullet, \alpha_\bullet) : \overline{R}^{b_{n-t}} \xrightarrow{\alpha_{n-t}} \overline{R}^{b_{n-t-1}} \rightarrow \dots \xrightarrow{\alpha_1} \overline{R}^{b_0} \rightarrow 0$  with  $H_0(P_\bullet) = \text{Ext}_{\overline{R}}^{n-t}(M, \overline{R})$  and a map  $\Psi_\bullet : P_\bullet \rightarrow L_\bullet^*$ .

such that  $\Psi_\bullet$  induces the injection  $c$  in (1) and the mapping cone of  $\Psi_\bullet$  is a free resolution of  $\text{Im } \phi_{n-t+1}^*$ . We have the following commutative diagram

$$\begin{array}{ccccccccccc}
P_\bullet : & \longrightarrow & \overline{R}^{b_{n-t+1}} & \xrightarrow{\alpha_{n-t+1}} & \overline{R}^{b_{n-t}} & \longrightarrow & \overline{R}^{b_{n-t-1}} & \cdots & \longrightarrow & \overline{R}^{b_1} & \longrightarrow & \overline{R}^{b_0} & \longrightarrow & 0 \\
& & & & \downarrow \Psi_{n-t} & & \downarrow \Psi_{n-t-1} & & & \downarrow \Psi_1 & & \downarrow \Psi_0 & & \\
L_\bullet^* : & 0 & \longrightarrow & \overline{R}^{s_0^*} & \longrightarrow & \overline{R}^{s_1^*} & \cdots & \longrightarrow & \overline{R}^{s_{n-t-1}^*} & \xrightarrow{\phi_{n-t}^*} & \overline{R}^{s_{n-t}^*} & \longrightarrow & 0
\end{array} \quad (2)$$

Applying  $\text{Hom}(-, \overline{R})$  to (2) we obtain the following commutative diagram of exact complexes

$$\begin{array}{ccccccccccc}
\longrightarrow & \overline{R}^{s_{n-t}} & \longrightarrow & \overline{R}^{s_{n-t-1}} & \longrightarrow & \cdots & \longrightarrow & \overline{R}^{s_1} & \longrightarrow & \overline{R}^{s_0} & \longrightarrow & M & \longrightarrow & 0 \\
& \downarrow \Psi_0^* & & \downarrow \Psi_1^* & & & & \downarrow \Psi_{n-t-1}^* & & \downarrow \Psi_{n-t}^* & & \downarrow \Psi & & \\
0 \longrightarrow & \overline{R}^{b_0^*} & \longrightarrow & \overline{R}^{b_1^*} & \longrightarrow & \cdots & \longrightarrow & \overline{R}^{b_{n-t-1}^*} & \xrightarrow{\alpha_{n-t}^*} & \overline{R}^{b_{n-t}^*} & \longrightarrow & Q & \longrightarrow & 0
\end{array} \quad (3)$$

where  $Q = \text{Coker } \alpha_{n-t}^*$  and  $\Psi : M \rightarrow Q$  is induced by  $\Psi_{n-t}^*$ . Since  $\text{grade Ext}_{\overline{R}}^i(M, \overline{R}) \geq i$ ,  $1 \leq i \leq n-t$ , the bottom row of (3) provides a minimal free resolution of  $Q$  and hence  $\text{pd}_{\overline{R}} Q = n-t$ . Moreover, by construction,  $\text{Ext}_{\overline{R}}^i(Q, \overline{R}) \xrightarrow{\sim} \text{Ext}_{\overline{R}}^i(M, \overline{R})$  for  $i > 0$ .

Next we want to prove that  $\Psi : M \rightarrow Q$  in (3) is injective.

Let  $\overline{F}_\bullet = F_\bullet \otimes \overline{R}$ ,  $S_i = \text{syzy}_R^i(M)$  and  $\overline{S}_i = S_i / \mathbf{x} S_i$ . Since  $\mathbf{x}$  is generated by an  $R$ -sequence of length  $t$  in  $\text{ann}_R M$ , we have  $\text{Tor}_t^R(M, \overline{R}) \xleftarrow{\sim} M$  and  $\text{Tor}_i^R(M, \overline{R}) = 0$  for  $i > t$ . Tensoring the exact sequence  $0 \rightarrow S_t \rightarrow R^{r_{t-1}} \rightarrow S_{t-1} \rightarrow 0$  by  $\overline{R}$  we obtain an exact sequence

$$0 \rightarrow M \xrightarrow{j} \overline{S}_t \rightarrow \overline{R}^{r_{t-1}} \rightarrow \overline{S}_{t-1} \rightarrow 0.$$

Let  $\theta_\bullet : L_\bullet \rightarrow \overline{F}_\bullet$  be a lift of  $j : M \hookrightarrow \overline{S}_t$ . We have the following commutative diagram:

$$\begin{array}{ccccccccccc}
L_\bullet : & \longrightarrow & \overline{R}^{s_{n-t}} & \longrightarrow & \overline{R}^{s_{n-t-1}} & \longrightarrow & \cdots & \longrightarrow & \overline{R}^{s_0} & \longrightarrow & 0 \\
& & \downarrow \theta_{n-t} & & \downarrow \theta_{n-t-1} & & & & \downarrow \theta_0 & & \\
\overline{F}_\bullet : & 0 \longrightarrow & \overline{R}^{r_n} & \longrightarrow & \overline{R}^{r_{n-1}} & \longrightarrow & \cdots & \longrightarrow & \overline{R}^{r_t} & \longrightarrow & \overline{R}^{r_{t-1}} & \longrightarrow & \cdots
\end{array} \quad (4)$$

The mapping cone of  $\{\theta_\bullet\}$  in (4) is a free resolution of  $\text{Im } \overline{d}_t$  over  $\overline{R}$  and  $\theta_0$  induces the isomorphism  $\tilde{\theta}_0 : M \xrightarrow{\sim} \text{Tor}_t^R(M, \overline{R})$  via  $j : M \hookrightarrow \overline{S}_t$ .

Applying  $\text{Hom}(-, \overline{R})$  to (4) the following commutative diagram is obtained:

$$\begin{array}{ccccccccccc}
(\overline{F}_\bullet^*) : & \longrightarrow & \overline{R}^{r_{t-1}^*} & \xrightarrow{\overline{d}_t^*} & \overline{R}^{r_t^*} & \xrightarrow{\overline{d}_{t+1}^*} & \cdots & \longrightarrow & \overline{R}^{r_{n-1}^*} & \xrightarrow{\overline{d}_n^*} & \overline{R}^{r_n^*} & \longrightarrow & \text{Ext}_R^n(M, \overline{R}) & \longrightarrow & 0 \\
& & & \downarrow \theta_0^* & & & & & \downarrow \theta_{n-t-1}^* & & \downarrow \theta_{n-t}^* & & \downarrow & & \\
(L_\bullet^*) : & 0 & \longrightarrow & \overline{R}^{s_0^*} & \longrightarrow & \cdots & \longrightarrow & \overline{R}^{s_{n-t-1}^*} & \longrightarrow & \overline{R}^{s_{n-t}^*} & \longrightarrow & G & \longrightarrow & 0
\end{array} \tag{5}$$

Since the mapping cone of (4) is a free resolution of  $\text{Im } \overline{d}_t$  over  $\overline{R}$  and  $\text{Ext}_R^i(M, \overline{R}) = 0$  for  $i > n - t$ ,  $\theta_{n-t}^*$  induces an isomorphism  $\text{Ext}_R^n(M, \overline{R}) \xrightarrow{\sim} \text{Ext}_R^{n-t}(M, \overline{R})$ . Hence the inclusion  $\text{Ext}_R^n(M, \overline{R}) \hookrightarrow G$  in (5) can be identified with  $c : \text{Ext}_R^{n-t}(M, \overline{R}) \hookrightarrow G$  in (1).

Let  $(H_\bullet, \delta_\bullet)$  denote the mapping cone of  $\Psi_\bullet$  in (2). By construction this is a free resolution of  $\text{Im } \phi_{n-t+1}^*$ . Let  $\eta_\bullet : L_\bullet^* \rightarrow H_\bullet$  ( $\eta_i : \overline{R}^{s_i^*} \rightarrow \overline{R}^{b_{n-i-t-1}} \oplus \overline{R}^{s_i^*}$ ),  $\gamma_\bullet : H_\bullet \rightarrow P_\bullet(-1)$  ( $\gamma_{n-j} : \overline{R}^{b_{n-j}} \oplus \overline{R}^{s_{j-t-1}^*} \rightarrow \overline{R}^{b_{n-j}}$ ) denote the corresponding inclusion and projection maps respectively. Then  $\eta_\bullet \cdot \theta_\bullet^* : \overline{F}_\bullet^* \rightarrow H_\bullet$  lifts the composite  $\eta \cdot c : \text{Ext}_R^{n-t}(M, \overline{R}) \xrightarrow{c} G \xrightarrow{\eta} \text{Im } \phi_{n-t+1}^*$ . Since  $\eta \cdot c = 0$  and  $(H_\bullet, \delta_\bullet)$  is a free resolution of  $\text{Im } \phi_{n-t+1}^*$ ,  $\eta_\bullet \cdot \theta_\bullet^*$  is homotopic to 0. Hence there exists homotopy maps  $h_\bullet : \overline{F}_\bullet^* \rightarrow H_\bullet$ ,  $h_j : \overline{R}^{r_j^*} \rightarrow \overline{R}^{b_{n-j}} \oplus \overline{R}^{s_{j-t-1}^*}$ , for  $t \leq j \leq n$ , such that

$$\delta_{n-j} \cdot h_j + h_{j+1} \cdot \overline{d}_{j+1}^* = \eta_{j-1} \cdot \theta_{j-1}^*. \tag{6}$$

Let  $\beta_\bullet = \gamma_\bullet \cdot h_\bullet$ ,  $\{\beta_j = \gamma_{n-j} \cdot h_j\}$ . Then  $\beta'_\bullet = \{(-1)^{j+1} \beta_j\} : \overline{F}_\bullet^* \rightarrow P_\bullet$  is a map of complexes. Consider  $\Psi_\bullet \cdot \beta'_\bullet : \overline{F}_\bullet^* \rightarrow L_\bullet^*$ .

For  $1 \leq j \leq n$ , let  $k_j = \pi_2 \cdot h_j : \overline{R}^{r_j^*} \rightarrow \overline{R}^{s_{j-t-1}^*}$ , where  $\pi_2 : \overline{R}^{b_{n-j}} \oplus \overline{R}^{s_{j-t-1}^*} \rightarrow \overline{R}^{s_{j-t-1}^*}$  is the projection on the second component. It can be checked that, for  $t \leq j \leq n$ ,

$$\theta_{j-1}^* - (-1)^{n-j-1} \Psi_{n-j} \cdot \beta_j = \phi_{j-1}^* \cdot k_j + k_{j+1} \cdot \overline{d}_{j+1}^*; \tag{7}$$

i.e.  $\theta_\bullet^*$  and  $\Psi_\bullet \cdot \beta'_\bullet$  are homotopic.

The commutative diagrams below are provided to clarify (6) and (7) for  $t = 1$ .

$$\begin{array}{c}
\begin{array}{ccccccc}
\overline{F}_\bullet^* & : & \dots & \longrightarrow & \overline{R}^{r_j^*} & \xrightarrow{\overline{d}_{j+1}^*} & \overline{R}^{r_{j+1}^*} \longrightarrow \dots \longrightarrow \overline{R}^{r_{n-1}^*} \longrightarrow \overline{R}^{r_n^*} \longrightarrow 0 \\
& & & & \downarrow \theta_{j-1}^* & \downarrow \theta_j^* & \downarrow \theta_{j+1}^* \\
& & & & \nearrow h_j & \nearrow h_{j+1} & \nearrow h_n \\
L_\bullet^* & : & \dots & \longrightarrow & \overline{R}^{s_{j-2}^*} & \longrightarrow & \overline{R}^{s_{j-1}^*} \longrightarrow \overline{R}^{s_j^*} \longrightarrow \dots \longrightarrow \overline{R}^{s_{n-t-1}^*} \longrightarrow \overline{R}^{s_{n-1}^*} \longrightarrow 0 \\
& & & & \downarrow \eta_{j-2} & \downarrow \eta_{j-1} & \downarrow \\
H_\bullet & : & \dots & \longrightarrow & \overline{R}^{b_{n-j}} \oplus \overline{R}^{s_{j-2}^*} & \xrightarrow{\delta_{n-j}} & \overline{R}^{b_{n-j-1}} \oplus \overline{R}^{s_{j-1}^*} \longrightarrow \dots \longrightarrow \overline{R}^{b_0} \oplus \overline{R}^{s_{n-t-1}^*} \longrightarrow \overline{R}^{s_{n-1}^*} \\
& & & & \downarrow \gamma_{n-j} & \downarrow \gamma_{n-j-1} & \downarrow \\
P_\bullet(-1) & : & \dots & \longrightarrow & \overline{R}^{b_{n-j}} & \xrightarrow{\alpha_{n-j}} & \overline{R}^{b_{n-j-1}} \longrightarrow \dots \longrightarrow \overline{R}^{b_0} \longrightarrow 0
\end{array} \\
\delta_{n-j} = (\alpha_{n-j}, (-1)^{n-j-1} \Psi_{n-j} + \phi_{j-1}^*)
\end{array}$$

$$\begin{array}{c}
\begin{array}{ccccccc}
\overline{F}_\bullet^* & : & \dots & \longrightarrow & \overline{R}^{r_j^*} & \longrightarrow & \overline{R}^{r_{j+1}^*} \\
& & & & \downarrow \beta_j & \downarrow \beta_{j+1} & \downarrow \\
& & & & \nearrow k_j & \nearrow k_{j+1} & \nearrow \\
P_\bullet & : & \dots & \longrightarrow & \overline{R}^{b_{n-j}} & \longrightarrow & \overline{R}^{b_{n-j-1}} \\
& & & & \downarrow \Psi_{n-j} & \downarrow \Psi_{n-j-1} & \downarrow \\
& & & & \nearrow \theta_{j-1}^* & \nearrow \theta_j^* & \nearrow \\
L_\bullet^* & : & \longrightarrow & \overline{R}^{s_{j-2}^*} & \longrightarrow & \overline{R}^{s_{j-1}^*} & \longrightarrow \overline{R}^{s_j^*}
\end{array}
\end{array}$$

Hence  $\{\theta_\bullet^*\}$  and  $\{\Psi_\bullet \beta'_\bullet\}$  define identical maps (modulo  $\pm$  sign) on homologies of  $F_\bullet^*$  and  $L_\bullet$  and consequently for the homologies of their respective duals. If  $\tilde{\Psi} : M \rightarrow H^{n-t}(P_\bullet^*)$ ,  $\tilde{\beta} : H^{n-t}(P_\bullet^*) \rightarrow \text{Tor}_t^R(M, \overline{R})$  and  $\tilde{\theta}_0 : M \xrightarrow{\sim} \text{Tor}_t^R(M, \overline{R})$  denote the maps induced by  $\Psi_{n-t}^*$ ,  $\beta_t^*$  and  $(\theta_0^*)^* = \theta_0$  respectively, then  $\tilde{\theta}_0 = \tilde{\beta} \cdot \tilde{\Psi}$  (modulo a sign). Since  $\tilde{\theta}_0$  is an isomorphism,  $\Psi = (\text{inclusion of } H^{n-t}(P_\bullet^*) \hookrightarrow Q) \cdot \tilde{\Psi} : M \hookrightarrow Q$  is injective.

Let  $\text{Coker } \psi = T$ ; then  $0 \rightarrow M \xrightarrow{\Psi} Q \rightarrow T \rightarrow 0$  is exact. Since the mapping cones of  $\Psi_\bullet^*, \Psi_\bullet$  are free resolutions of  $T$ ,  $\text{Im } \phi_{n-t+1}^*$  respectively and  $\text{Ext}_{\overline{R}}^i(M, \overline{R}) \xrightarrow{\sim} \text{Ext}_{\overline{R}}^i(Q, \overline{R})$  for  $i > 0$ , it follows that  $\text{Ext}_{\overline{R}}^i(T, \overline{R}) = 0$  for  $i > 0$  i.e.,  $\text{Ext}_R^i(T, R) = 0$  for  $i > t$ . Since  $\text{pd}_R M < \infty$ , we have  $\text{pd}_R T < \infty$  and hence  $\text{pd}_R T = t$ . This completes our proof.



**1.3 Corollary.** *Let  $R$  be a local ring and let  $M$  be a finitely generated module of finite projective dimension with  $\text{pd}_R M > 1$  and  $\text{grade}_R M > 0$ . Given any non-zero-divisor  $x \in \text{ann}_R M$ ,  $M$  can be imbedded in a finitely generated module  $Q$  of finite projective dimension over  $R/xR$  in such a way that if  $(G_\bullet, \gamma_\bullet)$ ,  $(F_\bullet, d_\bullet)$  are minimal free resolutions of  $M$  and  $Q$  over  $R$  respectively and  $\phi_\bullet : G_\bullet \rightarrow F_\bullet$  is a lift of  $i : M \hookrightarrow Q$ , then  $\phi_\bullet$  induces an isomorphism between  $(G_\bullet, \gamma_\bullet)_{i \geq 2}$  and  $(F_\bullet, d_\bullet)_{i \geq 2}$ ,  $\phi_1(G_1)$  is either a summand of or isomorphic to  $F_1$  and  $\phi_0$  is an injection. Moreover,  $\text{syzy}_R^1(M) \oplus R^t \simeq \text{syzy}_R^1(Q)$  for some  $t \geq 0$ .*

*Proof.* By the above theorem we have an exact sequence of  $R/xR$  modules

$$0 \rightarrow M \xrightarrow{i} Q \xrightarrow{\eta} T \rightarrow 0$$

where  $\text{pd}_{R/xR} Q < \infty$ ,  $\text{pd}_R T = 1$ .

The proof of the corollary now follows directly either by constructing a minimal free resolution of  $Q$  from minimal free resolutions of  $M$  and  $T$  or by extracting a minimal free resolution of  $M$  from the mapping cone of  $\eta_\bullet : F_\bullet \rightarrow L_\bullet$ , where  $F_\bullet, L_\bullet$  are minimal free resolutions of  $Q$  and  $T$  respectively. For details of such basic constructions the reader is referred to [Br-H].

**Remark.** Since  $\text{grade}_R T = \text{pd}_R T = r$ ,  $T$  is perfect. It can be easily checked from the above construction that  $\text{grade}_{\overline{R}} Q = 0$  and since  $\text{pd}_{\overline{R}} Q < \infty$ ,  $\text{support}_{\overline{R}} Q = \text{Spec}(\overline{R})$ . Moreover, the strong intersection conjecture ([H1], [P-S]) is valid for both  $Q$  and  $T$ . For details on this observations we refer the the reader to section 4, chapter II in [P-S].

## SECTION 2

**2.1 Lemma.** *Let  $(R, m)$  be a local ring of dimension of  $n$ . Assume that the order ideal conjecture is valid for local rings of dimension  $(n - 1)$ . Then for the validity of the order ideal conjecture on  $R$  it is enough to prove the validity of the assertion for the first syzygies of modules of finite projection. In particular, for cyclic modules of finite projective dimension over  $R$ , it is enough to prove that every minimal generator of any ideal in  $R$  of finite projective dimension over  $R$  is a non-zero-divisor in  $R$ .*

*Proof.* Let  $(F_\bullet, d_\bullet)$  be a minimal free resolution of  $M$  where  $F_i = R^{r_i}$  for  $i \geq 0$ . Let  $S_i$  denote the  $i^{th}$  syzygy of  $M$  for  $i \geq 1$  and let  $\beta$  be a minimal generator of  $S_i$  for  $i > 1$ . Then  $\beta = d_i(e)$  for some free generator  $e$  of  $F_i$  and we have  $\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_{r_{i-1}} \end{pmatrix} \in R^{r_{i-1}}$ . Let  $J$  denote the ideal generated by  $a_1, \dots, a_{r_{i-1}}$ . Let  $x$  be a non-zero-divisor on  $R$ ; then  $x$  is a non-zero-divisor on  $S_1$ . Let  $\bar{R} = R/xR$ ,  $\bar{S}_i = S_i/xS_i$ ,  $i \geq 1$ .  $\bar{S}_i$  is of finite projective dimension over  $\bar{R}$  and  $\bar{S}_i = Syz_{\bar{R}}^{i-1}(\bar{S}_1)$  for  $i > 1$ . By induction hypothesis,  $\text{grade}_{\bar{R}}(J + xR)/xR \geq (i-1)$ ; this implies  $\text{grade}_R J \geq (i-1)$ . Let  $y \in J$  be a non-zero-divisor on  $R$ . Let  $\tilde{R} = R/yR$ ,  $\tilde{S}_i = S_i/yS_i$  for  $i \geq 1$ . Then, again by arguing as above,  $\text{grade}_{\tilde{R}} J/(y) \geq (i-1)$ . Hence  $\text{grade}_R J \geq i$ . The second assertion now follows readily.

**2.2 Lemma.** *Let  $M$  be a finitely generated module of finite projective dimension over a local ring  $(R, m)$ . Suppose  $\text{rank}_R M = s$ . Then there exists a free submodule  $F = R^s$ , generated by a part of a minimal set of generators of  $M$ , such that  $M/F$  has positive grade.*

*Proof.* We induct on  $s = \text{rank}_R M$ . Since  $\text{pd}_R M < \infty$ , if  $s = 0$ , then for every associated prime  $P$  of  $R$ ,  $M_P = 0$  and hence  $\text{grade}_R M > 0$ . Now suppose that  $s > 0$ . Since  $\text{pd}_R M < \infty$ , by basic element method (lemma 2.1, [E-G4]) there exists a minimal generator  $\alpha$  of  $M$  such that image of  $\alpha$  is a part of a basis of  $M_P$  for every associated prime  $P$  of  $R$ . Hence we have a short exact sequence

$$0 \rightarrow R \rightarrow M \xrightarrow{\phi} M' \rightarrow 0 \quad (1 \rightarrow \alpha)$$

Then  $\text{pd}_R M' < \infty$  and  $\text{rank}_R M' = \text{rank}_R M - 1 = s - 1$ . By induction, there exists a free submodule  $F' = R^{s-1}$  of  $M'$ , generated by a part of a minimal set of generators of  $M'$ , such that a  $M'' = M'/F'$  has positive grade. We have the following short exact sequence:

$$0 \rightarrow F' \xrightarrow{i} M' \xrightarrow{\psi} M'' \rightarrow 0$$

Since  $F'$  is free we can lift  $i : F' \rightarrow M'$  to  $\eta : F' \rightarrow M$  such that  $\phi \cdot \eta = i$ . Let  $\theta = \psi \cdot \phi$ . It can be easily checked that  $\theta$  is onto and  $\text{Ker } \theta = R \oplus F'$ . Hence the lemma follows.

**2.3.** Our next theorem reduces the order ideal conjecture to the assertion that every minimal generator of a certain class of ideals of finite projective dimension must be a non-zero-divisor.

**Theorem.** *Let  $(R, m)$  be a local ring of dimension  $n$ . Assume that the order ideal conjecture is valid for local rings of dimension  $(n - 1)$ . Then for the validity of the order ideal conjecture over  $R$  it is enough to prove that every minimal generator of any ideal of grade 2, height 2 and of finite projective dimension over  $R$  is a non-zero-divisor in  $R$ .*

*Proof.* Let  $M$  be a finitely generated module of finite projective dimension over  $R$ . Due to Lemma 2.1, for the validity of the order ideal conjecture it is enough to consider minimal generators of  $S = \text{Syz}_R^1(M)$ . If  $\text{grade}_R M = 0$ , then  $\text{ann}_R M = 0$ ; Let  $s_0 = \text{rank}_R M$ . By the previous lemma there exists a free submodule  $F = R^{s_0}$  generated by a part of a minimal set of generators of  $M$  such that  $M' = M/F$  has positive grade. From the commutative diagram below

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & R^{s_0} & & R^{s_0} & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & S & \rightarrow & R^{r_0} & \xrightarrow{\eta} & M \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & S & \rightarrow & R^{r_0-s_0} & \rightarrow & M' \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array} \tag{1}$$

it follows that without any loss of generality we can assume  $\text{grade}_R M > 0$ . Let  $x$  be a non-zero-divisor in  $\text{ann}_R M$ . By corollary to theorem 1.2 we can assume that  $M$  has finite projective dimension over  $R/xR$  and  $\text{support}(M) = \text{support}(R/xR)$ . Let  $\overline{R} = R/xR$ ;  $\overline{S} = S/xS$ . By tensoring  $0 \rightarrow S \rightarrow R^{r_0} \rightarrow M \rightarrow 0$  with  $\overline{R}$  we obtain the following short exact sequences

$$0 \rightarrow M \xrightarrow{\theta} \overline{S} \rightarrow T \rightarrow 0, \quad 0 \rightarrow T \rightarrow \overline{R}^{r_0} \rightarrow M \rightarrow 0 \tag{2}$$

where  $T = \text{Syz}_{\overline{R}}^1(M)$ . If  $\gamma$  is a minimal generator of  $M$  and  $e \in R^{r_0}$ , a free generator of  $R^{r_0}$  is such that  $\eta(e) = \gamma$ , then it follows from chasing the commutative diagram obtained from multiplying the short exact sequence  $0 \rightarrow S \rightarrow R^{r_0} \rightarrow M \rightarrow 0$  by  $x$  that  $\theta(\gamma) = \text{Im}(xe)$  in  $\overline{S}$ . Let  $\alpha_i \in S$  denote the lifts of a minimal set of generators  $\overline{\alpha}_i$ ,  $1 \leq i \leq h$ , of  $T$ . By induction,  $\text{grade}_{\mathcal{O}_{T, \overline{R}}}(\overline{\alpha}_i) \geq 1$ ; hence  $\text{grade}_{\mathcal{O}_S}(\alpha_i) \geq 1$  for  $1 \leq i \leq h$ . Due to the exact sequences in (2) we obtain a minimal set of generators  $xe_1, \dots, xe_a, \alpha_1, \dots, \alpha_h$  of  $S$

where  $e_1, \dots, e_a$  form a part of a basis of  $R^{r_0}$ . If  $(F_\bullet, d_\bullet)$  ( $F_i = R^{r_i}$ ) denote a minimal free resolution of  $M$  over  $R$ , then there exist  $\tilde{e}_1, \dots, \tilde{e}_a, \dots, \tilde{e}_{a+j}, \dots$  a basis of  $R^{r_1}$  such that  $d_1(\tilde{e}_i) = xe_i$ ,  $1 \leq i \leq a$  and  $d_1(\tilde{e}_{a+j}) = \alpha_j$ ,  $1 \leq j \leq h$ . Any minimal generator of  $S$  is of the form

$$\sum_{i=1}^t c_i x e_i + \sum_{j=1}^h d_j \alpha_j$$

where at least one of  $c_i$ s or  $d_j$ s is a unit. If any  $d_j$  in the above expression is a unit then we are done by induction. Thus, it is easy to check that in order to show that for any minimal generator  $\beta$  of  $S$ ,  $\text{grade}_S(\beta) \geq 1$ , it is enough to consider  $\beta = xe - \sum \lambda_i \alpha_i$  where  $e \in \{e_1, \dots, e_a\}$ ,  $\sum \lambda_i \alpha_i \neq 0$  in  $T$ . Due to the 2nd exact sequence in (2), if any  $\lambda_i$  is a unit then we are done by induction. Hence we can assume that all  $\lambda_i$ s  $\in m$  in the above expression of  $\beta$ . Let  $t = \text{rank}_{\overline{R}} M$ . By arguing as in (1) we obtain the following commutative diagram

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & R^t & \xrightarrow{x} & R^t & \rightarrow & \overline{R}^t \\ & & \downarrow \psi & & \downarrow \phi_0 & & \downarrow \phi \\ 0 & \rightarrow & S & \rightarrow & R^{r_0} & \rightarrow & M \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & S' & \rightarrow & R^{r_0-t} & \rightarrow & M' \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array} \quad (3)$$

Let  $\{\bar{e}_j\}, 1 \leq j \leq t$  denote a basis of  $\overline{R}^t$  such that  $\gamma_j = \phi(\bar{e}_j)$ ,  $1 \leq j \leq t$ , in (3) is a part of a minimal set of generators of  $M$  (lemma 2.2). By commutativity of (3),  $\psi(e_j) = xe_j$ ,  $1 \leq j \leq t$ . Note that none of  $xe_j$  may be a part of  $\{xe_1, \dots, xe_a\}$  — part of a minimal basis of  $S$  mentioned above. We want to prove the following:

*Claim.* For any  $j$ ,  $1 \leq j \leq t$ , grade of the ideal generated by the entries of  $xe_j - \sum \lambda_i \alpha_i$  is  $\geq 1$  (here  $xe_j = \psi(e_j)$ ).

*Proof of the Claim.* Let  $\gamma_1, \dots, \gamma_t, \gamma_{t+1}, \dots, \gamma_{r_0}$  be a minimal set of generators of  $M$  where  $\gamma_j = \phi(\bar{e}_j)$ ,  $1 \leq j \leq t$  as above,  $t = \text{rank}_{\overline{R}} M$ . Then  $\text{grade}_{\overline{R}}(M') \geq 1$  i.e.  $\text{grade}_R(M') \geq 2$ . Let  $y \in \text{ann}_{\overline{R}} M'$  be a non-zero-divisor in  $\overline{R}$ ; then  $\forall k > t$ ,  $y\gamma_k = \sum_{j=1}^t a_{kj}\gamma_j$ ; hence  $ye_k - \sum a_{kj}e_j \in S$  for  $k > t$ . Let  $P \in \text{Ass}_R(R)$  and let  $q \in \text{Ass}_{\overline{R}}(\overline{R})$  containing  $P$ . By

construction  $q$  is an associated prime of  $M$  (recall  $\text{ann}_{\overline{R}} M = 0$ ); then  $M_q, T_q$  are free  $\overline{R}_q$  modules of rank  $t$  and  $r_0 - t$  respectively. Since  $\text{pd}_{R_q} M_q = 1$ ,  $S_q$  is also a free  $R_q$ -module. We have the following short exact sequence

$$0 \rightarrow M_q \rightarrow \overline{S}_q \rightarrow T_q \rightarrow 0 \quad 0 \rightarrow T_q \rightarrow \overline{R}_q^{r_0} \rightarrow M_q \rightarrow 0 \quad (4)$$

Let  $\overline{\beta}_k = \overline{e}_k - \sum_{j=1}^t \frac{\overline{a}_{kj}}{\overline{y}} \overline{e}_j$ . Then  $\{\overline{\beta}_k\}, t+1 \leq k \leq r_0$  is a basis of  $T_q$ . Let  $\beta_k = e_k - \sum_{j=1}^t \frac{a_{kj}}{y} e_j, t+1 \leq k \leq r_0$ . Then it follows from (4)  $\{xe_1, \dots, xe_t, \beta_{t+1}, \dots, \beta_{r_0}\}$  form a basis of  $S_q$ . In  $S_q$ , we have

$$\sum \lambda_i \alpha_i = \sum_{i=1}^t \frac{c_i}{b} x e_i + \sum_{k=t+1}^{r_0} \frac{d_k}{b} \beta_k, b \notin q \quad (5)$$

If  $\sum \lambda_i \alpha_i \notin q S_q$ , then  $\sum \lambda_i \overline{\alpha}_i \notin q T_q$  which implies that some  $d_k \notin q$ . Hence  $\text{Im}(x e_j - \sum \lambda_i \alpha_i)$  is a minimal generator of  $T_q$  which is free and thus  $x e_j - \sum \lambda_i \alpha_i \notin P R^{r_0}$ . Now suppose that  $\sum \lambda_i \alpha_i \in q S_q$  then all  $c_i, d_i \in q$  and  $x \nmid d_k$  for at least one  $k$  in (5). Hence  $x e_j - \sum \lambda_i \alpha_i = (1 - \frac{c_j}{b}) x e_j - \sum_{i \neq j} \frac{c_i}{b} x e_i - \sum \frac{d_{ik}}{b} \beta_k$ . Since  $1 - \frac{q}{b}$  is a unit in  $R_q$ ,  $x e_j - \sum \lambda_i \alpha_i$  is a minimal generator of  $S_q$ . Since  $S_q$  is a free  $R_q$  module,  $x e_j - \sum \lambda_i \alpha_i \notin P R_q^{r_0}$  and hence  $x e_j - \sum \lambda_i \alpha_i \notin P R^{r_0}$ . This completes the proof of our claim.

Due to the commutative diagram (3) and the above Lemma we can assume without any loss of generality  $\text{grade}_R M \geq 2$ . Now we appeal to the following result due to Smoke (Lemma 4.1, Th. 4.2, [Sm]). Given a finitely generated  $R$ -module  $M$  of grade  $\geq 2$ , we can construct an exact sequence

$$0 \rightarrow M \rightarrow R/I \rightarrow R/J \rightarrow 0 \quad (6)$$

where  $R/J$  has a filtration whose successive quotients are isomorphic to cyclic modules of the form  $R/(u, v)$  where  $\{u, v\}$  form an  $R$ -sequence. The proof in [Sm] shows that, by choosing the  $R$ -sequences of length 2 in  $m$  (annihilator of the corresponding module over  $R$ ), this exact sequence can be constructed in such a way that if  $\delta : S \rightarrow I$  denotes the corresponding map on first syzygies via (6), then  $\delta$  is surjective and  $\overline{\delta} : S/mS \rightarrow I/mI$  is an isomorphism. Since  $\text{pd}_R M < \infty$ ,  $\text{pd}_R R/I < \infty$  and it follows from the construction that height of  $I = \text{grade of } I = 2$ . Thus the proof of our theorem is complete.

**2.4.** Now we state the following Lemma.

**Lemma.** Let  $(R, m)$  be a local ring and let  $M$  be a finitely generated module of projective dimension  $n < \infty$ . Let  $N$  be an  $R$ -module such that  $\text{ann}_R M$  contains an  $N$ -sequence of length  $r$ . Then  $\text{Tor}_{n-i}^R(M, N) = 0$  for  $0 \leq i < r$ .

We leave the proof of this lemma to the reader.

**2.5.** In our next theorem we indicate the vanishing of Tor from an altogether different perspective.

**Theorem.** Let  $(R, m)$  be a local ring and let  $M$  be a finitely generated module of finite projective dimension over  $R$ . Let  $\text{pd}_R M = h$ . Then there exists an  $R$ -sequence  $x_1, \dots, x_h$  of length  $h$  such that for any  $R$ -module  $N$  for which  $x_1, \dots, x_h$  form an  $N$ -sequence,  $\text{Tor}_i^R(M, N) = 0$  for  $i > 0$ .

*Proof.* If  $\text{rank}_R M = s_0 > 0$ , by lemma 2.2, there exists a free submodule  $F = R^{s_0}$  generated by a part of a minimal set of generators of  $M$  such that  $M' = M/F$  has positive grade. From the commutative diagram

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
& & & R^{s_0} & & R^{s_0} & \\
& & & \downarrow & & \downarrow & \\
0 & \rightarrow & S & \rightarrow & R^{r_0} & \rightarrow & M \rightarrow 0 \\
& & \wr & & \downarrow & & \downarrow & \\
0 & \rightarrow & S & \rightarrow & R^{r_0-s_0} & \rightarrow & M' \rightarrow 0 \\
& & & \downarrow & & \downarrow & \\
& & & 0 & & 0 & 
\end{array} \tag{1}$$

it follows that  $S = \text{Syz}_R^1(M) = \text{Syz}_R^1(M')$ . If  $\text{rank}_R M = 0$ , then  $M' = M$ .

Let  $x_1 \in \text{ann}_R M'$  be a non-zero-divisor on  $R$  and let  $R_1 = R/x_1 R$ ,  $S_1 = S/x_1 S$ . By theorem 1.2 we have an exact sequence of  $R_1$ -modules

$$0 \rightarrow M' \rightarrow M_1 \rightarrow V_1 \rightarrow 0 \tag{2}$$

such that  $\text{pd}_{R_1} M_1 < \infty$  ( $= \text{pd}_R M - 1$ ) and  $\text{pd}_R V_1 = 1$ . It is also clear from Corollary 1.3 that  $S(M_1) = \text{Syz}_R^1(M_1) = S \oplus R^{t_1} = \text{Syz}_R^1(M') \oplus R^{t_1} = \text{Syz}_R^1(M) \oplus R^{t_1}$  for some  $t_1 \geq 0$ . Tensoring the short exact sequence  $0 \rightarrow S(M_1) \rightarrow R^{\ell_1} \rightarrow M_1 \rightarrow 0$  with  $R_1$ , we obtain the following short exact sequence

$$0 \rightarrow M_1 \rightarrow S(M_1) \otimes R_1 \rightarrow T_1 \rightarrow 0, \quad 0 \rightarrow T_1 \rightarrow R_1^{\ell_1} \rightarrow M_1 \rightarrow 0. \tag{3}$$

Here  $T_1 = \text{Syz}_{R_1}^1(M_1)$ . Recall that  $\text{pd}_{R_1} M_1 < \infty$  and  $\text{support}(M_1) = \text{Spec } R_1$ . Now we start with  $M_1$  over  $R_1$  and repeat the process described in (1), (2) and (3). We continue this process  $(h-2)$  times and obtain an  $R$ -sequence  $x_1, \dots, x_i$ ,  $1 \leq i \leq h-1$ , modules  $M_i$  of finite projective dimension over  $R_i = R/(x_1, \dots, x_i)$  and short exact sequences

$$0 \rightarrow M'_{i-1} \rightarrow M_i \rightarrow V_i \rightarrow 0 \quad (2i)$$

$$0 \rightarrow F_i \rightarrow M_i \rightarrow M'_i \rightarrow 0 \quad (1i)$$

where  $\text{pd}_{R_i} M_i = \text{pd}_R M - i$ ,  $\text{support}(M_i) = \text{support}(R_i)$ ,  $\text{pd}_{R_{i-1}} V_i = 1$ ,  $M'_{i-1} = M_{i-1}/F_{i-1}$ ,  $F_{i-1}$  a free  $R_{i-1}$ -module as constructed in (1),  $M'_i$  is a module over  $R_{i+1}$ . We also have short exact sequences of  $R_i$ -modules

$$0 \rightarrow M_i \rightarrow S(M_i) \otimes R_i \rightarrow T_i \rightarrow 0, \quad 0 \rightarrow T_i \rightarrow R_i^{\ell_i} \rightarrow M_i \rightarrow 0. \quad (3i)$$

We note that  $\text{grade } M_i = i$  and  $\text{pd}_R M_i = h$ . Let  $x_h \in m$  be such that  $\text{im}(x_h)$  in  $R_{h-1}$  is a non-zero-divisor contained in  $\text{ann}_{R_{h-1}} M'_{h-1}$ . Then  $x_1, \dots, x_h$  form an  $R$ -sequence and this is our required sequence.

Since  $\text{pd}_R M'_{h-1} = h$ , if  $N$  is an  $R$ -module such that  $x_1, \dots, x_h$  form an  $N$ -sequence, then it follows from the above lemma that  $\text{Tor}_i^R(M'_{h-1}, N) = 0$  for  $i > 0$ . Now it follows from the above short exact sequences starting from  $M'_{h-1}$  and tracing back to  $M$  that  $\text{Tor}_j^R(M, N) = 0$  for  $j \geq 1$ . If  $\text{grade}_R M = r > 0$  and  $x_1, \dots, x_r$  is an  $R$ -sequence contained in  $\text{ann}_R M$ , we start with  $Q$ —an  $\overline{R} (= R/(x_1, \dots, x_r))$  module as in Theorem 1.2 and construct an  $\overline{R}$ -sequence  $x_{r+1}, \dots, x_h$  by the above method. Then  $x_1, \dots, x_h$  form an  $R$ -sequence satisfying the required vanishing property of  $\text{Tor}$ .

Since  $\text{pd}_R M < \infty$ , if  $\text{grade}_R M = 0$  then  $\text{ann}_R M = 0$  and hence  $\text{rank}_R M > 0$ . Thus we are back to diagram (1).

**2.6 Corollary 1.** *Let  $(R, m)$  be a local ring and let  $M$  be a finitely generated module of finite projective dimension  $h$  over  $R$ . Let  $x_1, \dots, x_h$  be an  $R$ -sequence as mentioned in the above proposition. Suppose that for every  $P \in \text{Ass}_R(R)$ , there exists an  $R/P$ -module  $N$  such that  $x_1, \dots, x_h$  form an  $N$ -sequence and  $N \neq mN$ . Then, for every minimal generator  $\beta$  of  $S_1 = \text{Syz}_R^1(M)$ ,  $\text{grade } \mathcal{O}_{S_1}(\beta) \geq 1$ .*

*Proof.* Let  $(F_\bullet, d_\bullet)$  be a minimal resolution of  $M$  over  $R$ . If possible let  $\text{grade } \mathcal{O}_{S_1}(\beta) = 0$ . Then there exists an associated prime  $P$  such that  $\mathcal{O}_{S_1}(\beta) \subset P$ . Let  $\overline{R} = R/P$  and  $\overline{F}_\bullet = F_\bullet \otimes R/P$ . Consider the sequence

$$\overline{F}_2 \xrightarrow{\overline{d}_2} \overline{F}_1 \xrightarrow{\overline{d}_1} \overline{F}_0 \rightarrow 0.$$

Since  $\mathcal{O}_{S_1}(\beta) \subset P$ ,  $\overline{d}_1(\beta) = 0$ . This implies that for any  $\overline{R}$ -module  $N$ ,  $\text{Tor}_1^R(M, N) \neq 0$ . However, by hypothesis, there exists an  $\overline{R}$ -module  $N$  such that  $x_1, \dots, x_h$  form an  $N$ -sequence. Then, by the above proposition, we have  $\text{Tor}_j^R(M, N) = 0$  for  $j > 0$ , which leads to a contradiction. Hence  $\text{grade } \mathcal{O}_{S_1}(\beta) \geq 1$ .

**2.7 Corollary 2.** *Let  $(R, m)$  be a local ring and  $M$  be a finitely generated module of finite projective dimension  $h$ . Let  $x_1, \dots, x_h$  be an  $R$ -sequence as mentioned in the above proposition. If  $R$  has mixed characteristic  $p > 0$ , we assume that  $p, x_1, \dots, x_h$  form a part of a system of parameters of  $R$ . Then for every minimal generator  $\beta$  of  $S_1 = \text{Syz}_R^1(M)$ ,  $\text{grade } \mathcal{O}_{S_1}(\beta) \geq 1$ .*

*Proof.* If possible, let  $\text{grade } \mathcal{O}_{S_1}(\beta) = 0$ . Then there exists an associated prime  $P$  of  $R$  such that  $\mathcal{O}_{S_1}(\beta) \subset P$ . If  $R$  is equicharacteristic then there exists a big Cohen-Macaulay  $R/P$ -module  $N$  such that  $x_1, \dots, x_h$  form a regular  $N$ -sequence. If  $R$  has mixed characteristic  $p > 0$ , then there exists a big Cohen-Macaulay  $R/(P + pR)$ -module  $N$  such that  $x_1, \dots, x_h$  form a regular  $N$ -sequence. Hence we are done by Corollary 1.

**Remark** Due to the existence of Cohen-Macaulay algebras over local domains of dimension less than or equal to three ([H3]) it can be checked from the above arguments that finitely generated modules of projective dimension less than or equal to three satisfy the order ideal conjecture.

**2.8 Lemma.** *Let  $(R, m, K = R/m)$  be an equicharacteristic complete local ring of dimension  $d$  and let  $x_1, \dots, x_d$  be a system of parameters of  $R$ . Let  $M$  be a big Cohen-Macaulay  $R$ -module such that  $\mathbf{x}M \neq M$  and  $x_1, \dots, x_d$  form a maximal  $M$ -sequence. Then  $\widehat{M} =$  the  $m$ -adic completion of  $M$  is a flat  $K[[x_1, \dots, x_d]]$ -module.*

*Proof.* Let  $S = K[[x_1, \dots, x_d]]$ . Then  $S$  is a complete power series ring in  $d$  variables,  $R$  is a module finite extension of  $S$  and  $\mathbf{x} = \{x_1, \dots, x_d\}$  form a regular system of parameters



of  $S$ . Moreover  $\mathbf{x}$  is  $\widehat{M}$ -regular (Th. 8.5.1 [B-H]) and  $\widehat{M}$  is a balanced big Cohen-Macaulay module (Cor. 8.5.3, [Br-H]). Hence  $\psi : \frac{\widehat{M}}{\mathbf{x}\widehat{M}}[X_1, \dots, X_d] \rightarrow \bigoplus_{n=0}^{\infty} \frac{\mathbf{x}^n \widehat{M}}{\mathbf{x}^{n+1} \widehat{M}}$  is an isomorphism.

Since  $\mathbf{x}$  is a regular system of parameters of  $S$ ,  $K[X_1, \dots, X_d] \simeq \bigoplus_{n=0}^{\infty} \frac{\mathbf{x}^n S}{\mathbf{x}^{n+1} S}$ . Hence

$\frac{\mathbf{x}^n S}{\mathbf{x}^{n+1} S} \otimes_K \widehat{M}/\mathbf{x}\widehat{M} \simeq \frac{\mathbf{x}^n \widehat{M}}{\mathbf{x}^{n+1} \widehat{M}}$  and  $\frac{\widehat{M}}{\mathbf{x}\widehat{M}}$  is a non-null vector spacing over  $K = S/\mathbf{x}S$ . Thus it follows, by Th. 1, §5.2 in [Bou], that  $\widehat{M}$  is  $S$ -flat.

**2.9 Theorem.** (Foxby, [F]) *Let  $(R, m, K)$  be an equicharacteristic complete local ring and let  $N$  be a finitely generated module of finite projective dimension over  $R$ . Let  $I$  be an ideal of height  $h$  and let  $M$  be a big Cohen-Macaulay module over  $R/I$ . Let  $\widehat{M}$  be the  $m$ -adic completion of  $M$ . Then  $\mathrm{Tor}_i^R(\widehat{M}, N) = 0$  for  $i > h$ .*

*Proof.* For a proof we refer the reader to ([E-G4]) or ([F]) where the existence of a big Cohen-Macaulay  $R/I$ -module  $Q$  such that  $\mathrm{Tor}_i^R(Q, N) = 0$ , for  $i > h$  has been demonstrated (it was first proved by Foxby). In these proofs, it was required that such a  $Q$  be free over a certain complete regular local ring  $S$  contained in  $R/I$  such that  $R/I$  is a module-finite extension of  $S$ . By Lemma 2.8 the completion  $\widehat{M}$  of any big Cohen-Macaulay  $R/I$  module is flat over certain complete regular subrings of  $R/I$ . And this flatness is enough to ensure the validity of arguments provided in theorem 1.11 in ([E-G4]) or in ([F]) for proving our assertion.

**2.10.** Next we recall Shimomoto's theorem.

**Theorem** (Th. 5.3, [Shi]). *Let  $(R, m)$  be a complete local domain of mixed characteristic  $p > 0$ . Then there exist some system of parameters  $p, x_2, \dots, x_d$  of  $R$  and an almost Cohen-Macaulay quasi-local algebra  $B$  over  $R^+$ , the integral closer of  $R$  in algebraic closer of the field of fractions of  $R$ , in the sense that*

1.  $(p, x_2, \dots, x_d)B \neq B$ ,
2.  $x_2, \dots, x_d$  form a regular sequence on  $B/pB$ , and
3.  $p$  is not nilpotent in  $B$  and the ideal  $(0 : p)_{\mathbf{B}}$  is annihilated by  $p^{\infty}$  for any rational  $\infty > 0$ .

Actually Shimomoto's construction shows that such a  $B$  can be constructed for any

system of parameters of the form  $p, x_2, \dots, x_d$  of  $R$ .

**Definition.** An almost Cohen-Macaulay algebra  $B$  as above is called balanced if  $B/pB$  is a balanced Cohen-Macaulay algebra.

**2.11.** Now we are ready to prove our final theorem.

**Theorem.** Let  $(R, m)$  be a local ring of mixed characteristic  $p > 0$ . We assume that either  $p$  is nilpotent or  $p$  is a non-zero-divisor in  $R$ . Let  $M$  be a finitely generated module of finite projective dimension over  $R$  and  $\beta$  be a minimal generator of  $S_i$ , the  $i^{\text{th}}$  syzygy of  $M$  (minimal), for  $i > 0$ . We have the following:

- a) If  $p$  is nilpotent, the order ideal conjecture is valid on  $R$ ;
- b) if  $pM = 0$ , then  $\text{grade of } \mathcal{O}_{S_1}(\beta) \geq 1$ ; and
- c) Suppose that balanced or complete almost Cohen-Macaulay algebras exist over complete local domains. If  $pM = 0$  and  $p \in m - m^2$ , then  $\text{grade of } \mathcal{O}_{S_i}(\beta) \geq i, \forall i \geq 1$ .
- d) Assume that every element in  $m - m^2$  is a non-zero-divisor and that the order ideal conjecture is valid over  $R/xR$  for any  $x \in m - m^2$ . If  $\text{ann}_R M \cap (m - m^2) \neq \phi$  or  $\text{depth}_R M > 0$  then  $\text{grade } \mathcal{O}_{S_i}(\beta) \geq i, \forall i \geq 1$ .

*Proof.*

a) If  $p$  is nilpotent, the proof follows immediately by similar arguments as in (Th. 2.4, [E-G2]) due to the existence of big Cohen-Macaulay modules on equicharacteristic local domains  $R/P$  for every prime ideal  $P$  in  $\text{Spec } R$ . One may also use Lemma 9.1.8 from [Br-H] to prove the assertion.

b) Since  $p$  is a non-zero-divisor on  $R$ , by the corollary of Theorem 1.2 we can assume  $\text{pd}_{R/pR} M < \infty$ . We write  $\overline{R} = R/pR$ . Consider the short exact sequence

$$0 \rightarrow S_1 \xrightarrow{g} R^{r_0} \xrightarrow{\eta} M \rightarrow 0. \quad (1)$$

Tensoring this sequence with  $\overline{R}$  we obtain the following exact sequences

$$\mathcal{O} \rightarrow M \xrightarrow{j} \overline{S}_1 \xrightarrow{b} T_1 \rightarrow 0, \quad (2)$$

$$\mathcal{O} \rightarrow T_1 \xrightarrow{\gamma} \overline{R}^{r_0} \xrightarrow{\overline{\eta}} M \rightarrow 0 \quad (3)$$

where  $T_1 = \text{Syz}_{\overline{R}}^1(M)$ ,  $\overline{S}_1 = S_1 \otimes_R \overline{R}$ . For any minimal generator  $\vartheta$  of  $M$  we have  $\vartheta = \eta(e)$ ,  $e$  being a minimal generator of  $R^{r_0}$  and  $j(\vartheta) = \text{class of } pe \text{ in } \overline{S}_1$ . Recall that if  $I$  is an ideal

of  $R$  such that  $\text{grade}_R I = 0$ , then for any non-zero-divisor  $x$  in  $R$ ,  $\text{grade}_{\overline{R}}(I + xR)/xR$  is also 0. Hence, for any minimal generator  $\beta$  of  $S_1$ , if  $b(\overline{\beta})$  is a minimal generator of  $T_1$  then our assertion follows due to the validity of the order ideal conjecture on equicharacteristic local rings. If  $b(\overline{\beta}) = 0$ , then  $\overline{\beta} \in j(M)$  i.e.  $\overline{\beta} = j(\vartheta)$  where  $\vartheta$  is a minimal generator of  $M$ . Let  $\{\overline{\alpha}_j\}_{1 \leq j \leq t}$  be a minimal set of generators of  $T_1$  and  $\{\alpha_j\}$  denote a lift of  $\{\overline{\alpha}_j\}$  in  $S_1$ . Then there exists a basis  $e_1, \dots, e_i, \dots, e_{r_0}$  of  $R^{r_0}$  such that  $\{pe_1, \dots, pe_i, \alpha_j s\}$  form a minimal set of generators of  $S_1$ . Moreover, in order to show that for any minimal generator  $\beta$  of  $S_1$ ,  $\text{grade}_{\mathcal{O}_{S_1}}(\beta) \geq 1$ , it is enough to take  $\beta = pe - \sum_{j=1}^t \lambda_j \alpha_j$ , where  $e \in \{e_1, \dots, e_i\}$ ,  $\lambda_j s \in m$  and  $\sum \lambda_j \overline{\alpha}_j \neq 0$  in  $T_1$ .

If possible let  $\text{grade}_{\mathcal{O}_{S_1}}(\beta) = 0$  i.e.  $\mathcal{O}_{S_1}(\beta) \subset P$ , for some  $P \in \text{Ass}_R(R)$ . Let  $\text{pd}_R M = h$ ; then  $\text{pd}_{\overline{R}} M = (h - 1)$ . By theorem 2.5, corresponding to  $M$  over  $\overline{R}$ , there exists an  $\overline{R}$  sequence  $\overline{x}_1, \dots, \overline{x}_{h-1}$  satisfying the assertion mentioned in theorem (2.5). Let  $\tilde{R} = R/PR$  and let  $\tilde{p} = im(p)$  in  $\tilde{R}$  and  $\tilde{x}_i = im(x_i)$  in  $\tilde{R}$ ,  $1 \leq i \leq h - 1$ . Then  $\tilde{p}, \tilde{x}_1, \dots, \tilde{x}_{h-1}$  form a part of a system of parameters of  $\tilde{R}$ . By Corollary 2.7 there exists an almost Cohen-Macaulay  $\tilde{R}^+$  algebra  $B$  such that  $(\tilde{p}, \dots, \tilde{x}_{h-1})B \neq B$  and  $\tilde{x}_1, \dots, \tilde{x}_{h-1}$  form a regular sequence on  $B/pB$ . Then, by theorem 2.5 or by theorem 2.4 in [F], we have  $\text{Tor}_j^{\tilde{R}}(M, B/pB) = 0$  for  $j > 0$ .

We consider the following part of a minimal resolution  $F_\bullet$  of  $M$  over  $R$ :

$$\begin{array}{ccccccc} R^{r_2} & \xrightarrow{d_2} & R^{r_1} & \xrightarrow{d_1} & R^{r_0} & \rightarrow & 0 \\ \parallel & & \parallel & & \parallel & & \\ F_2 & \rightarrow & F_1 & \rightarrow & F_0 & \rightarrow & 0 \end{array}.$$

Let  $\tilde{F}_i = F_i \otimes \tilde{R}$ ,  $0 \leq i \leq 2$ . Tensoring the above sequence with  $\tilde{R}$  we obtain a sequence

$$\tilde{F}_2 \xrightarrow{\tilde{d}_2} \tilde{F}_1 \xrightarrow{\tilde{d}_1} \tilde{F}_0 \rightarrow 0$$

where  $\tilde{\beta} = im(\beta)$  in  $\tilde{F}_0 = 0$ .

Tensoring the above sequence with  $B$  and writing  $F_{iB} = F_i \otimes B$  we have

$$F_{2B} \xrightarrow{d_{2B}} F_{1B} \xrightarrow{d_{1B}} F_{0B} \rightarrow 0 \quad (4)$$

where  $\beta_B = im\tilde{\beta}$  in  $F_{0B} = 0$ . Hence

$$pe_B = \sum \lambda_j \alpha_{jB} \quad \text{in} \quad B^{r_0} = F_{0B}, \quad (5)$$

where  $\alpha_{jB} = im(\alpha_j \otimes 1_B)$  in  $F_{0B}$ ,  $\alpha_j \otimes 1_B \in S_1 \otimes B$ . Since  $\text{Tor}_j^{\overline{R}}(M, B/pB) = 0$ , for  $j > 0$ , tensoring (2) and (3) with  $B/pB$  over  $\overline{R}$  we obtain the following short exact sequences:

$$0 \rightarrow M \otimes B/pB \xrightarrow{j \otimes 1_{B/pB}} \overline{S}_1 \otimes B/pB \rightarrow T_1 \otimes B/pB \rightarrow 0 \quad (6)$$

and

$$\mathcal{O} \rightarrow T_1 \otimes B/pB \rightarrow (B/pB)^{r_0} \rightarrow M \otimes B/pB \rightarrow 0. \quad (7)$$

Let  $\overline{\alpha_{jB}}$  = the image of  $(\overline{\alpha_j} \otimes 1_{B/pB})$  in  $T_1 \otimes B/pB$  in  $(B/pB)^{r_0} = im \alpha_{jB}$  in  $(B/pB)^{r_0}$ . Due to (5) and (7), we have  $\sum \lambda_j \overline{\alpha_{jB}} = 0$  in  $(B/pB)^{r_0}$  and hence  $\sum \lambda_j (\overline{\alpha_j} \otimes 1_{\overline{B}}) = 0$  in  $T_1 \otimes B/pB$ . Since  $\lambda_j s \in m$ , this implies, due to the exact sequence (6) and the definition of  $j$  in (2), that

$$\sum \lambda_j (\alpha_j \otimes 1_B) = \sum a_i (p e_i \otimes 1_B) + p \left( \sum b_i (p e_i \otimes 1_B) + \sum \mu_i (\alpha_i \otimes 1_B) \right) \quad (8)$$

in  $S_1 \otimes B$ ,  $a_i s \in m_B$ , where  $m_B$  = maximal ideal of  $B$ .

Hence, we have from (5) and (8).

$$p e_B = \sum (a_i + p b_i) p e_{iB} + p \sum \mu_i \alpha_{iB}$$

in  $B^{r_0}$  i.e.

$$p \left\{ e_B - \left[ \sum (a_i + p b_i) e_{iB} + \sum \mu_i \alpha_{iB} \right] \right\} = 0. \quad (*)$$

Since  $a_i$ , entries of  $\alpha_{i\beta} \in m_B$ ,  $e_B - [\sum (a_i + p b_i) e_{iB} + \sum \mu_i \alpha_{iB}]$  is a free generator of  $B^{r_0}$  and hence  $p$  can not annihilate it. Thus  $(*)$  leads to a contradiction. Hence grade  $\mathcal{O}_{S_1}(\beta)$  must be  $\geq 1$ .

c) We assume  $p \in m - m^2$  and  $pM = 0$ . Let  $F_\bullet : 0 \rightarrow R^{r_n} \xrightarrow{d_n} R^{r_{n-1}} \rightarrow \dots \rightarrow R^{r_1} \xrightarrow{d_1} R^{r_0} \rightarrow 0$  be a minimal free resolution of  $M$  over  $R$  and let  $P_\bullet : 0 \rightarrow \overline{R}^{s_{n-1}} \rightarrow \dots \rightarrow \overline{R}^{s_1} \rightarrow \overline{R}^{s_0} \rightarrow 0$  be a minimal free resolution of  $M$  over  $\overline{R}(= R/pR)$ . Shamash ([Sha]) has shown that  $P_\bullet$  can be obtained from  $F_\bullet$  via the homotopy maps  $h_\bullet : F_\bullet \rightarrow F_\bullet(+1)$  induced by the 0-map on  $M$  due to multiplication by  $p$ . Actually each  $F_i$ , for  $i > 0$ , can be decomposed into two parts:  $F_i = F'_i \oplus F''_i$ , where  $h_i(F''_i) = 0$ , every free generator  $e'$  of  $F'_{i-1}$  is such that  $e = h_{i-1}(e')$  is a free generator of  $F''_i$  and  $d_i(e) = p e' - h_{i-2} d_{i-1}(e')$ . Moreover, it follows from Shamash's theorem that I) there exists  $\phi_\bullet : P_\bullet \rightarrow \overline{F_\bullet(+1)}(\overline{F_\bullet} = F_\bullet \otimes_R \overline{R})$ , where

$\phi_0$  induces the inclusion map:  $M = \text{Tor}_1^R(M, R/p) \xrightarrow{j} \overline{S}_1$  and II)  $\phi_\bullet$  induces a splitting on each component of  $P_\bullet$  and  $P_\bullet(+1)$  can be extracted from the mapping cone of  $\phi_\bullet$ . Let  $S_i = \text{Syz}_R^i(M)$ ,  $T_i = \text{Syz}_{\overline{R}}^i(M)$  and  $\overline{S}_i = S_i \otimes \overline{R}$ . This leads to the following short exact sequences for  $i > 1$ :

$$0 \rightarrow T_{i-1} \xrightarrow{\psi} \overline{S}_i \xrightarrow{\eta} T_i \rightarrow 0, \quad (1')$$

$$0 \rightarrow T_i \rightarrow \overline{R}^{r_{i-1}} \rightarrow T_{i-1} \rightarrow 0 \quad (2')$$

where  $\psi$  is induced by  $\phi_\bullet$  and  $\psi(\text{Im } \overline{e}') = \overline{pe - h_{i-2}d_{i-1}(e')}$  is a minimal generator of  $\overline{S}_i$  due to the splitting property of  $\phi_\bullet$ . Let  $\gamma = pe' - h_{i-2}d_{i-1}(e')$  and

$$\overline{\gamma} = \overline{pe' - h_{i-2}d_{i-1}(e')}. \quad (3')$$

Then  $\gamma, \overline{\gamma}$  are minimal generators of  $S_i$  and  $\overline{S}_i$  respectively.

**Claim:**  $\text{grade } \mathcal{O}_{S_i}(\gamma) \geq i$ .

*Proof of the claim.* If possible let  $P$  be a prime ideal of height  $(i-1)$  containing  $\mathcal{O}_{S_i}(\gamma)$ . Since  $e' \in F'_{i-1}$  and  $h_{i-2}(F_{i-2}) = F''_{i-1}$ , it follows that  $p \in P$ . Let  $p, x_2, \dots, x_{i-1}$  be a maximal  $R$ -sequence contained in  $\mathcal{O}_{S_i}(\gamma)$ —we denote it by  $\mathbf{x}$ . Then  $\text{Tor}_j^R(M, R/\mathbf{x}) = 0$  for  $j \geq i$ . Let  $R' = R/\mathbf{x}$ ,  $S'_i = S_i \otimes R'$  etc. We have an exact sequence

$$0 \rightarrow S'_i \rightarrow R'^{r_{i-1}} \rightarrow S'_{i-1} \rightarrow 0.$$

Then  $S'_{i-1}$  is a module of finite projective dimension over  $R'$  and  $S'_i = \text{Syz}_{R'}^1(S'_{i-1})$  has a minimal generator  $\gamma' = im \gamma$  in  $S'_i$  such that  $\mathcal{O}_{S'_i}(\gamma')$  has grade 0 in  $R'$ . This contradicts part b) of our theorem and hence the claim is established.

Let  $\{\overline{\alpha}_j\}$  form a minimal set of generators of  $T_i$  and let  $\{\alpha_j\}$  denote their lifts in  $S_i$ . Since characteristic of  $\overline{R} = p > 0$ , by Evans-Griffith theorem ([E-G2]),  $\text{grade}_{\overline{R}}(\mathcal{O}_{T_i}(\overline{\alpha}_j)) \geq i$ . Since  $(1') \otimes R/m$  is exact, due to the above claim, for the purpose of proving our theorem it would be enough to establish that for every minimal generator  $\beta$  of  $S_i$  of the form  $\beta = \gamma - \sum \lambda_j \alpha_j$ ,  $\text{grade } \mathcal{O}_{S_i}(\beta) \geq i$ , where  $\overline{\gamma} = \psi(im \overline{e}')$  for some free generator  $e'$  for  $F_{i-1}$  and  $\lambda_j s \in m$ . If possible let  $\mathcal{O}_{S_i}(\beta) \subset P$ —a prime ideal of height  $(i-1)$ . If  $p \in P$ , there exists a maximal Cohen-Macaulay  $R/P$ -algebra  $((R/P)^+ \text{ algebra}) \overline{B}$ . If  $p \notin P$ , then,

by assumption, there exists a balanced almost Cohen-Macaulay  $R/P$ -algebra  $((R/P)^+$  algebra)  $B$  such that  $p$  is not nilpotent on  $B$  and  $\overline{B} = B/pB$  is maximal Cohen-Macaulay algebra over  $R/(P + pR)$ . Hence, in either case, by Theorem 2.4 in [F],  $\text{Tor}_j^{\overline{R}}(M, \overline{B}) = 0$  for  $j \geq i$ . Tensoring (1') and (2') with  $\overline{B}$  we get the following exact sequences

$$\mathcal{O} \rightarrow T_{i-1} \otimes \overline{B} \xrightarrow{\psi \otimes 1_{\overline{B}}} \overline{S}_i \otimes \overline{B} \rightarrow T_i \otimes \overline{B} \rightarrow 0 \quad (6')$$

and

$$\mathcal{O} \rightarrow T_i \otimes \overline{B} \rightarrow \overline{B}^{r_{i-1}} \rightarrow T_{i-1} \otimes \overline{B} \rightarrow 0. \quad (7')$$

Let  $e$  be a free generator of  $F_i$  such that  $d_i(e) = \beta$ .

From  $F_\bullet$ , we consider the exact sequence

$$F_{i+1} \xrightarrow{d_{i+1}} F_i \xrightarrow{d_i} F_{i-1}.$$

Let  $\tilde{R} = R/P$ ,  $\tilde{F}_i = F_i \otimes_R \tilde{R}$ . Tensoring the above sequence with  $\tilde{R}$ , we obtain a complex

$$\tilde{F}_{i+1} \xrightarrow{\tilde{d}_{i+1}} \tilde{F}_i \xrightarrow{\tilde{d}_i} \tilde{F}_{i-1}$$

where  $\tilde{d}_i(\tilde{e}) = 0$ . Let  $e_B = e \otimes 1_B$ . Then in the complex  $F_{i+1} \otimes B \rightarrow F_i \otimes B \rightarrow F_{i-1} \otimes B$ , we have  $d_{iB}(e_B) = 0$ . Let  $\gamma_B = \text{im}(\gamma \otimes 1_B)$ ,  $\alpha_{jB} = \text{im}(\alpha_j \otimes 1_B)$  in  $B^{r_{i-1}} = F_{i-1} \otimes B$ . Then

$$d_{iB}(e_B) = 0 \Rightarrow \gamma_B - \sum \lambda_j \alpha_{jB} = 0. \quad (5')$$

Let  $\overline{\alpha}_{jB} = \text{Im } \alpha_{jB}$  in  $\overline{B}^{r_{i-1}}$ . Due to (5') and (6') we have  $\sum \lambda_j \overline{\alpha}_{jB} = 0$  in  $\overline{B}^{r_{i-1}}$ , hence  $\sum \lambda_j (\overline{\alpha}_j \otimes 1_B) = 0$  in  $T_i \otimes \overline{B}$ . Since  $\lambda_j s \in m$ , this implies, due to the exact sequence (6') and definition of  $\psi$  in (1'), that

$$\sum \lambda_j (\alpha_j \otimes 1_B) = \sum a_i (\gamma_i \otimes 1_B) + p \left[ \left( \sum b_i (\gamma_i \otimes 1_B) + \sum \mu_j (\alpha_j \otimes 1_B) \right) \right] \quad (8')$$

in  $S_i \otimes B$ ,  $a_i s \in m_B$ . Hence from (5') we have

$$\gamma_B = \sum (a_i + pb_i) (\gamma_{iB}) + p \sum \mu_j \alpha_{jB}$$

in  $B^{r_{i-1}}$ . Since  $\gamma_B = p e'_B - h_{i-2} d_{i-1}(e'_B)$  and  $\gamma_{iB}s$  also have similar expressions, comparing the  $e'_B$ th co-ordinate in  $B^{r_{i-1}}$ , we obtain from above

$$p \left[ e'_B - \sum (a_i + pb_i) e_{B'} - \sum \delta_j \right] = 0 \quad (**)$$

where  $\delta_j = e'_B{}^{\text{th}}$  co-ordinate of  $\mu_j \alpha_{jB}$ . Since  $a_i, \delta_j \in m_B$ , the term within brackets in (\*\*) is a free generator of  $B^{r_{i-1}}$  and hence  $p$  cannot annihilate it. Thus (\*\*) leads to a contradiction. Hence  $\text{grade } \mathcal{O}_{S_i}(\beta) \geq i$  and our proof is complete.

d) We assume that every element in  $m - m^2$  is a non-zero-divisor and for any such element  $x$  the order ideal conjecture is valid for  $R/xR$ . Let  $M$  be a finitely generated module of finite projective dimension such that either  $\text{ann}_R M \cap (m - m^2) \neq \phi$  or  $\text{depth}_R M > 0$ .

First let us assume that  $\text{ann}_R M \cap (m - m^2) \neq \phi$ . Let  $x \in \text{ann}_R M \cap (m - m^2)$  and let  $\overline{R} = R/xR$ . Since  $\text{pd}_R M < \infty$  and  $x \in m - m^2$ ,  $\text{pd}_{\overline{R}} M < \infty$ . By hypothesis  $M$  satisfies the order ideal conjecture as an  $\overline{R}$ -module. Let  $(F_\bullet, d_\bullet), (P_\bullet, S_\bullet)$  denote minimal free resolutions of  $M$  over  $R$  and  $\overline{R}$  respectively; let  $S_i = \text{Syz}_R^i(M)$  and  $T_i = \text{Syz}_{\overline{R}}^i(M)$ . Arguing as in the proof of part b) of the theorem ((1), (2), (3) etc.) we construct a minimal set of generators  $\{xe_1, \dots, xe_i, \alpha_j s\}$  such that  $\{e_1, \dots, e_j\}$  form a part of a basis of  $F_0$ ,  $\overline{\alpha}_j (= im \alpha_j \in T_1)$  form a minimal set of generators of  $T_1$ . In order to show that for any minimal generators  $\beta$  of  $S_1$ ,  $\text{grade } \mathcal{O}_{S_1}(\beta) \geq 1$ , due to inductive hypothesis, it is enough to take  $\beta = xe - \sum \lambda_i \alpha_i$ ,  $e \in \{e_1, \dots, e_i\}, \lambda_i \in m$ . Then  $\beta_e =$  the  $e$ -th co-ordinate of  $\beta = x - \sum \lambda_i \alpha_{ie}$ . Since  $\lambda_i s \in m$ ,  $\alpha_{ie} s \in m$  and  $x \in m - m^2$ , we have  $\beta_e \neq 0$ ,  $\beta_e \in m - m^2$  and hence by assumption  $\beta_e$  is a non-zero-divisor in  $R$ . Thus the conclusion follows for  $i = 1$ .

Now consider  $i > 1$ . Arguing as in part c) above we see from (1'), (2'), (3') etc. that it is enough to consider a minimal generator  $\beta$  of  $S_i$  of the form  $\beta = \gamma - \sum \lambda_i \alpha_i$  where  $\gamma = xe' - h_{i-2}h_{i-1}(e') = d_i(e), e = h_{i-1}(e'), e'$  a minimal generator of  $F'_{i-1}$  and  $\overline{\alpha}_j (= im \alpha_j \in T_i)$  form a minimal set of generators of  $T_i$ . Similar arguments as in part c) show that  $\text{grade of } \mathcal{O}_{S_i}(\gamma) \geq i$  and by hypothesis,  $\text{grade } \mathcal{O}_{T_i}(\overline{\alpha}_j) \geq i$ . Let  $J$  denote the ideal generated by entries of  $\beta$ . Recall that  $F_{i-1} = F'_{i-1} \oplus F''_{i-1}$ ,  $e'$  is a minimal generator of  $F'_{i-1}$ ,  $h_{i-2}d_{i-1}(e') \in F''_{i-1}$ . Let  $y =$  the  $e'$ -th co-ordinate of  $\beta = x - \sum \lambda_i \alpha_{ie'}$ ; since  $\lambda_i s \in m$  and  $\alpha_{ie'} s \in m$ ,  $y \in m - m^2$  and hence, by assumption,  $y$  is a non-zero-divisor in  $R$ . Let  $\overline{R} = R/yR$ ,  $\overline{S}_i = S_i/yS_i$  for  $i \geq 1$ ; then  $\overline{S}_i = \text{Syz}_{\overline{R}}^{i-1}(\overline{S}_1)$ . By hypothesis  $J/yR$  has  $\text{grade} \geq (i-1)$  in  $\overline{R}$ . This implies that  $\text{grade}_R J \geq i$  and our proof is complete.

Now assume  $\text{depth}_R M > 0$ . we can find  $x \in m - m^2$  such that  $x$  is a non-zero-divisor on  $M$ . Let  $\overline{R} = R/xR$ ,  $\overline{M} = M/xM$ . Since  $\text{pd}_{\overline{R}} M = \text{pd}_R M < \infty$ , by hypothesis,  $\overline{M}$  satisfies the order ideal conjecture over  $\overline{R}$ . And hence  $M$  satisfies the order ideal conjecture over

R.

**Corollary 1.** *Let  $(R, m)$  be a local ring of mixed characteristic  $p > 0$  such that  $p$  is a non-zero-divisor in  $R$ . Let  $I$  be an ideal of finite projective dimension over  $R$ . If  $p \in I$  or  $p$  is a non-zero-divisor on  $R/I$  then every minimal generator of  $I$  is a non-zero-divisor in  $R$ . In particular if  $P$  is a prime ideal of finite projective dimension over  $R$ , then every minimal generator of  $P$  is a non-zero-divisor.*

*Proof.* If  $p \in I$ , the proof follows from part b) of the above theorem; if  $p$  is a non-zero-divisor on  $R/I$ , the result follows from the validity of the order ideal conjecture on  $R/pR$  ([E-G2]).

**Corollary 2.** *Let  $(R, m)$  be a regular local ring of dimension  $n$  and assume that the order ideal conjecture is valid for regular local rings of dimension  $(n - 1)$ . If  $M$  is a finitely generated  $R$ -module such that either  $M$  is annihilated by a regular parameter or  $\text{depth}_R M > 0$ , then  $M$  satisfies the order ideal conjecture.*

Proof follows from part d) of the above theorem.

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